Problem 2.35

A particle of mass m and kinetic energy $E > 0$ approaches an abrupt potential drop V_0 (Figure 2.19).[54]

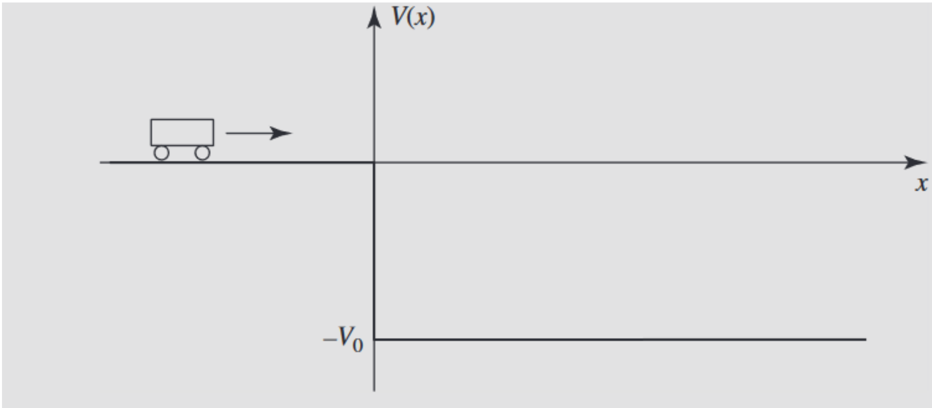

Figure 2.19: Scattering from a "cliff" (Problem 2.35).

(a) What is the probability that it will "reflect" back, if $E = V_0/3$? *Hint:* This is just like Problem 2.34, except that the step now goes *down*, instead of up.

(b) I drew the figure so as to make you think of a car approaching a cliff, but obviously the probability of "bouncing back" from the edge of a cliff is *far* smaller than what you got in (a)—unless you're Bugs Bunny. Explain why this potential does *not* correctly represent a cliff. *Hint:* In Figure 2.20 the potential energy of the car drops *discontinuously* to $-V_0$, as it passes $x = 0$; would this be true for a falling car?

(c) When a free neutron enters a nucleus, it experiences a sudden drop in potential energy, from $V = 0$ outside to around -12 MeV (million electron volts) inside. Suppose a neutron, emitted with kinetic energy 4 MeV by a fission event, strikes such a nucleus. What is the probability it will be absorbed, thereby initiating another fission? *Hint:* You calculated the probability of *reflection* in part (a); use $T = 1 - R$ to get the probability of transmission through the surface.

Solution

[54] For further discussion see P. L. Garrido, et al., *Am. J. Phys.* **79**, 1218 (2011).

Part (a)

Schrödinger's equation describes the time evolution of the wave function $\Psi(x,t)$.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

Here

$$V(x,t) = V(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ -V_0 & \text{if } x > 0 \end{cases}$$

models the drop in potential energy. Applying the method of separation of variables [assuming that $\Psi(x,t) = \psi(x)\phi(t)$] yields the following two ODEs.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + V(x) &= E \end{aligned}\right\}$$

The ODE in x is known as the time-independent Schrödinger equation (TISE).

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi.$$

Split up the TISE over the intervals that $V(x)$ is defined on.

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(-E)\psi, \quad x \leq 0 \qquad\qquad \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(-V_0 - E)\psi, \quad x > 0$$

For $E = V_0/3$ in particular,

$$\frac{d^2\psi}{dx^2} = -\frac{2mV_0}{3\hbar^2}\psi, \quad x \leq 0 \qquad\qquad \frac{d^2\psi}{dx^2} = -\frac{8mV_0}{3\hbar^2}\psi, \quad x > 0.$$

The general solution for $\psi(x)$ is then

$$\psi(x) = \begin{cases} A\exp\left(i\sqrt{\frac{2mV_0}{3\hbar^2}}x\right) + B\exp\left(-i\sqrt{\frac{2mV_0}{3\hbar^2}}x\right) & \text{if } x \leq 0 \\ F\exp\left(i\sqrt{\frac{8mV_0}{3\hbar^2}}x\right) + G\exp\left(-i\sqrt{\frac{8mV_0}{3\hbar^2}}x\right) & \text{if } x > 0 \end{cases}.$$

Solving the ODE in t yields $\phi(t) = e^{-iEt/\hbar}$, so the product solution $\Psi(x,t) = \psi(x)\phi(t)$ is a linear combination of plane waves travelling to the left and to the right. Assuming a plane wave is only incident from the left, $G = 0$.

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } x \leq 0 \\ Fe^{2ikx} & \text{if } x > 0 \end{cases}$$

The constant k has been introduced to make the formula more compact.

$$k = \sqrt{\frac{2mV_0}{3\hbar^2}}$$

By definition, the reflection coefficient is

$$R = \left|\frac{\text{reflected probability current}}{\text{incident probability current}}\right|,$$

where the probability current is

$$\begin{aligned}
J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \\
&= \frac{i\hbar}{2m}\left\{[\psi(x)e^{-iEt/\hbar}]\frac{\partial}{\partial x}[\psi^*(x)e^{iEt/\hbar}] - [\psi^*(x)e^{iEt/\hbar}]\frac{\partial}{\partial x}[\psi(x)e^{-iEt/\hbar}]\right\} \\
&= \frac{i\hbar}{2m}\left\{[\psi(x)e^{-iEt/\hbar}]\frac{d\psi^*}{dx}e^{iEt/\hbar} - [\psi^*(x)e^{iEt/\hbar}]\frac{d\psi}{dx}e^{-iEt/\hbar}\right\} \\
&= \frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx}\right).
\end{aligned}$$

For this formula of $\psi(x)$, the reflection coefficient is

$$\begin{aligned}
R &= \left|\frac{\frac{i\hbar}{2m}\left[(Be^{-ikx})\frac{d}{dx}(B^*e^{ikx}) - (B^*e^{ikx})\frac{d}{dx}(Be^{-ikx})\right]}{\frac{i\hbar}{2m}\left[(Ae^{ikx})\frac{d}{dx}(A^*e^{-ikx}) - (A^*e^{-ikx})\frac{d}{dx}(Ae^{ikx})\right]}\right| \\
&= \left|\frac{(Be^{-ikx})(ikB^*e^{ikx}) - (B^*e^{ikx})(-ikBe^{-ikx})}{(Ae^{ikx})(-ikA^*e^{-ikx}) - (A^*e^{-ikx})(ikAe^{ikx})}\right| \\
&= \left|\frac{2ikBB^*}{-2ikAA^*}\right| \\
&= \frac{|B|^2}{|A|^2}.
\end{aligned}$$

Require the wave function [and consequently $\psi(x)$] to be continuous at $x = 0$ to determine one of the constants.

$$\lim_{x\to 0^-}\psi(x) = \lim_{x\to 0^+}\psi(x): \quad A + B = F$$

Integrate both sides of the TISE with respect to x from $-\epsilon$ to ϵ, where ϵ is a really small number, to determine one more.

$$\begin{aligned}
\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}\,dx &= \int_{-\epsilon}^{\epsilon}\frac{2m}{\hbar^2}[V(x) - E]\psi(x)\,dx \\
\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} &= \int_{-\epsilon}^{0}\frac{2m}{\hbar^2}(-E)\psi(x)\,dx + \int_{0}^{\epsilon}\frac{2m}{\hbar^2}(-V_0 - E)\psi(x)\,dx \\
&= \frac{2m}{\hbar^2}(-E)\psi(0)\int_{-\epsilon}^{0}dx + \frac{2m}{\hbar^2}(-V_0 - E)\psi(0)\int_{0}^{\epsilon}dx \\
&= \frac{2m}{\hbar^2}(-E)\psi(0)\epsilon + \frac{2m}{\hbar^2}(-V_0 - E)\psi(0)\epsilon
\end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{0^-}^{0^+} = 0$$

It turns out that $\partial\Psi/\partial x$ is continuous at $x = 0$ as well.

$$\lim_{x\to 0^-} \frac{d\psi}{dx} = \lim_{x\to 0^+} \frac{d\psi}{dx} : \quad ik(A - B) = 2ikF$$

Substitute the formula for F and solve for B.

$$ik(A - B) = 2ik(A + B)$$

$$B = -\frac{1}{3}A$$

Therefore, the reflection coefficient (the probability that the mass will turn back at $x = 0$) is

$$R = \left(\frac{B}{A}\right)\left(\frac{B}{A}\right)^* = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^* = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{1}{9}.$$

Part (b)

The potential energy given by $V(x)$ drops discontinuously from 0 to $-V_0$ at $x = 0$, but this is not what happens for a car that drives off a cliff. Rather, the car's potential energy drops continuously from 0 to $-V_0$ in a linear fashion: $V_{\text{car}}(x) = -mgx$, where x is the vertical distance fallen.

Part (c)

A neutron with energy $E = 4$ MeV is a third of $V_0 = 12$ MeV, so the analysis from part (a) holds here. By definition, the transmission coefficient is

$$T = \left|\frac{\text{transmitted probability current}}{\text{incident probability current}}\right|,$$

where the probability current is

$$\begin{aligned}
J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \\
&= \frac{i\hbar}{2m}\left\{[\psi(x)e^{-iEt/\hbar}]\frac{\partial}{\partial x}[\psi^*(x)e^{iEt/\hbar}] - [\psi^*(x)e^{iEt/\hbar}]\frac{\partial}{\partial x}[\psi(x)e^{-iEt/\hbar}]\right\} \\
&= \frac{i\hbar}{2m}\left\{[\psi(x)e^{-iEt/\hbar}]\frac{d\psi^*}{dx}e^{iEt/\hbar} - [\psi^*(x)e^{iEt/\hbar}]\frac{d\psi}{dx}e^{-iEt/\hbar}\right\} \\
&= \frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx}\right).
\end{aligned}$$

For

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } x \leq 0 \\ Fe^{2ikx} & \text{if } x > 0 \end{cases},$$

the transmission coefficient is

$$\begin{aligned}
T &= \left|\frac{\frac{i\hbar}{2m}\left[(Fe^{2ikx})\frac{d}{dx}(F^*e^{-2ikx}) - (F^*e^{-2ikx})\frac{d}{dx}(Fe^{2ikx})\right]}{\frac{i\hbar}{2m}\left[(Ae^{ikx})\frac{d}{dx}(A^*e^{-ikx}) - (A^*e^{-ikx})\frac{d}{dx}(Ae^{ikx})\right]}\right| \\
&= \left|\frac{(Fe^{2ikx})(-2ikF^*e^{-2ikx}) - (F^*e^{-2ikx})(2ikFe^{2ikx})}{(Ae^{ikx})(-ikA^*e^{-ikx}) - (A^*e^{-ikx})(ikAe^{ikx})}\right| \\
&= \left|\frac{-4ikFF^*}{-2ikAA^*}\right| \\
&= 2\frac{|F|^2}{|A|^2}.
\end{aligned}$$

Use the continuity of the wave function and its first spatial derivative at $x = 0$.

$$\begin{aligned}
\lim_{x\to 0^-}\psi(x) &= \lim_{x\to 0^+}\psi(x): \quad A + B = F \\
\lim_{x\to 0^-}\frac{d\psi}{dx} &= \lim_{x\to 0^+}\frac{d\psi}{dx}: \quad ik(A - B) = 2ikF
\end{aligned}$$

Divide both sides of the second equation by ik and then add the respective sides to eliminate B.

$$2A = 3F$$

Solve for F.

$$F = \frac{2}{3}A$$

Therefore, the transmission coefficient (the probability that the neutron will enter the nucleus) is

$$T = 2\left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^* = 2\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^* = 2\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{8}{9}.$$

As expected, $R + T = 1$.